University Paris-Sud Exercice sheet 1 Representation theory

Let k be a field of characteristic $\neq 2$. Let R be a commutative k-algebra and let

$$\mathrm{Der}_k(R) := \{D \in \mathrm{End}_k(R) \quad | \quad D(fg) = D(f)g + fD(g) \text{ for all } f, g \in R\}.$$

1. Show that $\mathrm{Der}_k(R)$ is a sub-Lie algebra of $\mathrm{End}_k(R)$.
2. Compute $\mathrm{Der}_k(k[t])$ and $\mathrm{Der}_k(k[t, t^{-1}])$.
3. Let $f, g \in k[t]$, compute $[f\partial_t, g\partial_t]$. More generally prove that if $f, g \in R$ and $D, D' \in \mathrm{Der}_k(R)$ we have the equality $[fD, f'D'] = fg[D, D'] + fD(g)D' - gD'(g)D$.

Recall that an affine algebraic group G over k is the datum of a finitely generated k-algebra $H = \mathcal{O}(G)$ (the ring of functions on G) together with morphisms of k-algebras:

$$\Delta : H \longrightarrow H \otimes_k H, \epsilon : H \longrightarrow k, S : H \longrightarrow H,$$

subject to certain axioms. The most basic examples are the additive group

$$\mathbb{G}_a = (k, +)$$

and the multiplicative group

$$\mathbb{G}_m = (k^*, \times) = (k \setminus \{0\}, \times).$$

4. What are the quadruples $(\mathcal{O}(\mathbb{G}_a), \Delta, \epsilon, S)$ and $(\mathcal{O}(\mathbb{G}_m), \Delta, \epsilon, S)$?

Fix an algebraic group G and define the tangent space at the identity element e to be the k-vector space

$$\mathrm{Der}_k(H, k) := \{D \in \mathrm{Hom}_k(H, k) \quad | \quad D(fg) = D(f)\epsilon(g) + \epsilon(f)D(g) \text{ for all } f, g \in H\}.$$

Let

$$\mathcal{L}(G) := \{D \in \mathrm{Der}_k(H) \quad | \quad \Delta \circ D = (\mathrm{id} \otimes D) \circ \Delta\}$$

be the space of left-invariant vector fields. One checks that $\mathcal{L}$ is a sub-Lie algebra of $\mathrm{Der}_k(H)$ and that the following maps are linear isomorphisms, inverses of one another:

$$x \in \mathrm{Der}_k(H, k) \longmapsto (\mathrm{id} \otimes x) \circ \Delta \in \mathcal{L}(G),$$
$$X \in \mathcal{L}(G) \longmapsto \epsilon \circ X \in \mathrm{Der}_k(H, k).$$

In particular we can endow $\mathfrak{g} = \mathrm{Lie}(G) := \mathrm{Der}_k(H, k)$ with the structure of a Lie algebra via this isomorphism.

5. Compute $\mathrm{Lie}(\mathbb{G}_a)$ and $\mathrm{Lie}(\mathbb{G}_m)$.

Define $\mathcal{O}(SL_2) = \mathbb{C}[a, b, c, d]/(ad - bc - 1)$.

6. What is $(\mathcal{O}(SL_2), \Delta, \epsilon)$?
7. Compute $\mathfrak{sl}_2 = \mathrm{Lie}(sl_2)$.

Let $n \in \mathbb{N}$ and define

$$\mathcal{O}(GL_n) = \mathbb{C}\left[(a_{i,j})_{1 \le i,j \le n}\right]\left[\frac{1}{\det(a_{i,j})}\right] = \mathbb{C}\left[(a_{i,j})_{1 \le i,j \le n}\right]\left[D^{-1}\right]/(D^{-1}\det(a_{i,j}) - 1).$$

8. What is $(\mathcal{O}(GL_n), \Delta, \epsilon)$?
9. Compute $\mathfrak{gl}_2 = \mathrm{Lie}(GL_2)$.
10. *(If we have time)* Compute $\mathfrak{gl}_n$.