University Paris-Sud Exercice sheet 2 Representation theory

Let $\mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ be the 3-dimensional Lie algebra such that

$$[e, f] = -2f, [e, f] = h, [h, e] = 2e.$$

Let V be a $\mathfrak{sl}_2$-module. We say that V is a weight module if h acts semisimply (i.e. is diagonalizable) on V. For $\lambda \in \mathbb{C}$, we let V^λ be the eigenspace of h corresponding to λ.

1. Show that the sum $\sum_{\lambda \in \mathbb{C}} V^\lambda$ is direct. If x has weight λ then $e \cdot x$ (resp. $f \cdot x$) has weight $\lambda + 2$ (resp. $\lambda - 2$).

2. Show that $\mathfrak{sl}_2$ is simple.

We say that a nonzero vector $v \in V$ is maximal of weight $\lambda \in \mathbb{C}$ if $e \cdot x = 0$ and $h \cdot x = \lambda x$.

3. Find a maximal vector of $\mathfrak{sl}_2$, what is its weight ?

4. Let $v \in V$ be a primitive element of weight λ. For all $n \in \mathbb{N}$, let $v_n = \frac{f^n \cdot v}{n!}$ and $v_{-1} = 0$. Show that
   - $h \cdot v_n = (\lambda - 2n) v_n$
   - $f \cdot v_n = (n + 1) v_{n+1}$
   - $e \cdot v_n = (\lambda - n + 1) v_{n-1}$.

5. Show that only two cases may arise: either the elements (v_n) are linearly independant or the weight λ belongs to $\mathbb{N}$, the elements $v_0, ..., v_\lambda$ are linearly independant and $v_i = 0$ for all $i > \lambda$.

We take the following fact for granted: any finite dimensional $\mathfrak{sl}_2$-module is a weight module.

6. Show that every nonzero finite-dimensional $\mathfrak{sl}_2$-module V contains a primitive element v of weight $\lambda \in \mathbb{N}$.

7. Show that the vector subspace W of V with basis $v_0, ..., v_\lambda$ is stable under $\mathfrak{sl}_2$ and is irreducible.

8. Classify all finite dimensional simple $\mathfrak{sl}_2$-modules.

9. Let $\mathfrak{sl}_2$ act on the polynomial ring $\mathbb{C}[X, Y]$ by derivations via

$$f \mapsto Y \frac{\partial}{\partial X}, h \mapsto X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y}, e \mapsto X \frac{\partial}{\partial Y}.$$

   Show that this infinite dimensional representation is completely reducible and describe its irreducible summands.

For $\lambda \in \mathbb{C}$, define $M(\lambda)$ to be a $\mathbb{C}$-vector space with countable basis $\{v_0, v_1, v_2, ...\}$.

10. Show that the relations of question 4 make $M(\lambda)$ into a $\mathfrak{sl}_2$-module.

11. For what values of λ is $M(\lambda)$ irreducible ? When it is not, is it a direct sum of simple modules ? *Hint : notice that any $\mathfrak{sl}_2$-submodule of $M(\lambda)$ is infinite dimensional.*