University Paris-Sud Exercice sheet 3 Representation theory

Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{U}(\mathfrak{g})$ be its envelopping algebra. Denote by $(\mathcal{U}(\mathfrak{g})_{\leq n})_{n\geq 0}$ the PBW filtration. Recall that $\mathcal{U}(\mathfrak{g})$ is an almost-commutative filtered algebra, this means that for all $m, n \geq 0$ the map

$$[\cdot,\cdot] : (x \otimes y) \in \mathcal{U}(\mathfrak{g})_{\leq m} \otimes \mathcal{U}(\mathfrak{g})_{\leq n} \longmapsto (xy - yx) \in \mathcal{U}(\mathfrak{g})_{\leq n+m}$$

in fact lands in $\mathcal{U}(\mathfrak{g})_{n+m-1}$.

1. Describe $\mathcal{U}(\mathfrak{g})_{\leq 1}$ in terms of the Lie algebra $\mathfrak{g}$.

2. Show that $\mathcal{U}(\mathfrak{g})$ has no zero-divisors.

3. Show that the map
$$[\cdot,\cdot] : \mathcal{U}(\mathfrak{g})_{\leq 1} \otimes \mathcal{U}(\mathfrak{g})_{\leq n} \longrightarrow \mathcal{U}(\mathfrak{g})_{\leq n}$$
induces a $\mathfrak{g}$-module structure on $\mathcal{U}(\mathfrak{g})$. We call it the adjoint representation.

4. Assume that $\mathfrak{g} = \mathfrak{sl}_2$, does the previous structure makes $\mathcal{U}(\mathfrak{sl}_2)$ into a weight module ? Try and give two proofs, one with and one without computation.

5. Make $\mathcal{U}(\mathfrak{sl}_2)$ into a $\mathfrak{sl}_2$-module via the multiplication map:
$$\mathcal{U}(\mathfrak{sl}_2) \otimes \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{U}(\mathfrak{sl}_2).$$
Does this module structure makes $\mathcal{U}(\mathfrak{sl}_2)$ into a weight module ?

Define
$$C = \frac{1}{2}h^2 + ef + fe \in \mathcal{U}(\mathfrak{sl}_2)_{\leq 2}.$$

6. Show that C is a central element in $\mathcal{U}(\mathfrak{sl}_2)$.

7. What does the previous statement means in term of the structure of $\mathcal{U}(\mathfrak{sl}_2)$ as a $\mathfrak{sl}_2$-module for the adjoint representation ?

8. Let $\lambda \in \mathbb{N}$ and $L(\lambda)$ be the unique simple $\lambda + 1$-dimensional representation of $\mathfrak{sl}_2$. How does C acts on $L(\lambda)$? Same question with $M(\lambda)$ (see exercice sheet 2 for definitions).

Recall the following version of Schur's Lemma: let $\mathfrak{g}$ be a Lie algebra and V, W be two simple $\mathfrak{g}$-modules, then $\mathrm{Hom}_{\mathfrak{g}}(V, W) = 0$ if V and W are non-isomorphic and $\mathbb{C}$ if V and W are isomorphic.

9. Using Schur's Lemma, what can you say a priori on the action of the center of $\mathcal{U}(\mathfrak{g})$ on a simple $\mathfrak{g}$-module V ?

10. *(Bonus)* Describe $\mathcal{U}(\mathfrak{sl}_2)$ as a $\mathfrak{sl}_2$-module for the adjoint representation.