University Paris-Sud Exercice sheet 4 Representation theory

We are interested in the four infinite families of classical Lie algebras A_l, B_l, C_l, D_l ($l \geq 1$), these are Lie subalgebras of $\mathfrak{gl}_n$ for $n = l+1, 2l+1, 2l, 2l$ respectively. For $1 \leq i, j \leq n$, denote by $e_{i,j}$ be the corresponding elementary matrix in $\mathfrak{gl}_n$ and $M_{m,n}$ be the vector space of $m \times n$ matrices. Recall that

$$[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - \delta_{l,i} e_{k,j},$$
$$[e_{i,i}, e_{j,k}] = (\delta_{i,j} - \delta_{k,i}) e_{j,k}.$$

- Type A_l: the special linear Lie algebra $\mathfrak{sl}_{l+1}$ consisting of trace zero matrices.
- Type B_l: the special orthogonal Lie algebra of odd dimension

$$\mathfrak{so}_{2l+1} = \left\{ \begin{pmatrix} 0 & u & v \\ -{}^t v & a & b \\ -{}^t u & c & -{}^t a \end{pmatrix} \mid u, v \in M_{1,l},\ a, b, c \in M_l,\ {}^t b = -b,\ {}^t c = -c \right\}.$$

- Type C_l: the symplectic Lie algebra

$$\mathfrak{sp}_{2l} = \left\{ \begin{pmatrix} a & b \\ c & -{}^t a \end{pmatrix} \mid a, b, c \in M_{l,l},\ {}^t b = b,\ {}^t c = c \right\}.$$

- Type D_l: The special orthogonal Lie algebra of even dimension

$$\mathfrak{so}_{2l} = \left\{ \begin{pmatrix} a & b \\ c & -{}^t a \end{pmatrix} \mid a, b, c \in M_l,\ {}^t b = -b,\ {}^t c = -c \right\}.$$

1. Compute the dimension of the classical Lie algebras.
2. Show that $D_1 \simeq \mathbb{C}$, $A_1 \simeq B_1 \simeq C_1$.
3. *(Bonus)* Show that $B_2 \simeq C_2$, $A_3 \simeq D_3$. What about D_2 ?
4. Compute the root space decomposition of $\mathfrak{sl}_2$ and draw the root system of type A_1.
5. For each of the classical Lie algebras, check that a Cartan subalgebra $\mathfrak{h}$ is given by the subalgebra consisting of diagonal matrices.
6. Compute the dimension of the Cartan subalgebra $\mathfrak{h}$ (also known as the rank) and the number of roots in each case.

Recall that an invariant form is given by the trace.

7. Compute the root space decomposition of $\mathfrak{sl}_3$ and draw the root system of type A_2.
8. Compute the root space decomposition of $\mathfrak{so}_5$ and draw the root system of type B_2.
9. Compute the fundamental weights for the root system of type A_1, A_2 and B_2.