University Paris-Sud Exercice sheet 5 Representation theory

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be semisimple Lie algebras.

1. Show that as associative algebras

$$\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}') \simeq \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}').$$

2. If V (resp. V') is a finite dimensional $\mathfrak{g}$-module (resp. $\mathfrak{g}'$-module), show that the $\mathbb{C}$-vector space $V \otimes V'$ is naturally a $\mathfrak{g} \oplus \mathfrak{g}'$-module. We denote this $\mathfrak{g} \oplus \mathfrak{g}'$-module

$$V \boxtimes V'.$$

3. Show that if V and W are simple, then so is $V \boxtimes W$. *Hint: Use highest weight theory and show that if V is of highest weight λ and W if of highest weight μ then show that $V \boxtimes W$ has a highest weight vector of weight (λ, μ) and conclude.*

From now on we assume that $\mathfrak{g} = \mathfrak{g}'$.

4. Recall why $V \otimes W$ has a natural structure of $\mathfrak{g}$-module.

Recall that if V is a finite dimensional $\mathfrak{g}$-module then we define the formal character $\mathrm{ch}(V)$ to be the sum

$$\sum_{\lambda \in P} \dim(V_\lambda) e^\lambda \in \mathbb{Z}[P]$$

where $\mathbb{Z}[P]$ is the free $\mathbb{Z}$-module with basis given by the elements e^λ labelled by elements of the weight lattice $\lambda \in P$.

5. Compute the character of $V \otimes W$ in term of the characters of V and W.

Assume momentaneously that $\mathfrak{g} = \mathfrak{sl}_2$. Let $\lambda \in \mathbb{N}$ and $L(\lambda)$ be the finite dimensional simple module of highest weight λ.

6. Compute $\mathrm{ch}(L(\lambda))$.

7. Let $\lambda, \mu \in \mathbb{N}$. Is the sl_2-module $L_\lambda \otimes L_\mu$ simple ? If not, compute its decomposition as a direct sum of $\mathfrak{sl}_2$-modules.

8. Explain how to construct $L(1)$ very explicitely and how to obtain every other finite dimensional simple representation from it.

9. For which values of λ and μ does $L(\lambda) \otimes L(\mu)$ contains the trivial representation ? Try to generalize your result for an arbitrary semisimple Lie algebra.

10. For any semisimple Lie algebra $\mathfrak{g}$ and $\lambda, \mu \in P^+$ show that $L(\lambda + \mu)$ appears as a subrepresentation of $L(\lambda) \otimes L(\mu)$ with multiplicity one.

We now consider the Lie algebra $\mathfrak{sl}_3$ with Cartan $\mathfrak{h}$. Let $h_1 = e_{11} - e_{22}$ and $h_2 = e_{22} - e_{33}$ be a basis of $\mathfrak{h}$ and $\omega_1, \omega_2 \in \mathfrak{h}^*$ be the dual basis.

11. Show that ω_1 and ω_2 are the fundamental weights.

12. Show that the standard representation of $\mathfrak{sl}_3$ on $\mathbb{C}^3$ equals $L(\omega_1)$.

13. Show that the dual of the standard representation equals $L(\omega_2)$.

14. Explain how to obtain every finite dimensional simple module of $\mathfrak{sl}_3$ from these two.