University Paris-Sud Exercice sheet 6 Representation theory

1. *(Warm-up)* Let $\mathfrak{g}$ be a reductive finite-dimensional Lie algebra, is it true that any finite-dimensional representation of $\mathfrak{g}$ is semisimple ?

Let $\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}b_n \oplus \mathbb{C}C$ be the Lie algebra with basis $(b_n)_{n\in\mathbb{Z}}$ and C whose structure is given by the fact that C is central and the relation:

$$[b_m, b_n] = m\delta_{m+n,0}C.$$

1. Show that b_0 is central and that the $(b_{\pm n})_{n>0}$ commute with one another.[1]

2. Let M be a simple $\mathcal{H}$-module, show that b_0 and C must act via scalars.

3. Let M be a finite-dimensional simple $\mathcal{H}$-module, show that C must act by zero. *Hint: use the fact that the trace of a commutator is zero.*

From now on we assume that C acts by the identity. We say that a $\mathcal{H}$-module is smooth is for all $m \in M$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $b_n m = 0$. For all $\lambda \in \mathbb{C}$, define the Fock space $\mathcal{F}_\lambda$ of highest-weight λ to be the $\mathbb{C}$-vector space

$$\mathcal{F}_\lambda = \mathbb{C}[(x_{-n})_{n>0}]|\lambda\rangle$$

where b_0 acts by λ and for all $n > 0$, b_{-n} acts by multiplication by x_{-n} and b_n acts by $n\frac{\partial}{\partial x_{-n}}$.

4. Show that Fock spaces are smooth $\mathcal{H}$-modules.

5. Show that Fock spaces are simple $\mathcal{H}$-modules.

Consider $\mathcal{C}$ to be the full subcategory of smooth $\mathcal{H}$-modules such that b_0 acts semisimply and each $(b_n)_{n>0}$ acts locally nilpotently.

6. Show that Fock spaces belong to $\mathcal{C}$.

7. Show that giving a morphism of $\mathcal{H}$-modules from the Fock space $\mathcal{F}_\lambda$ to a module M amounts to giving a vector $m \in M$ such that $b_0 m = \lambda m$ and $b_n m = 0$ if $n > 0$.

We want to show that the Fock spaces are the only simple objects in the category $\mathcal{C}$.

8. Show that any nonzero object of $\mathcal{C}$ contains a Fock space.

9. Deduce that the Fock spaces are the only simple objects of $\mathcal{C}$.

In fact, the category $\mathcal{C}$ is even semisimple, this is not hard, but a bit long.

[1] The operators $(b_{-n})_{n>0}$ are called the creation operators and the $(b_n)_{n>0}$ are called the annihilation operators. The terminology will become clearer when we start computing in Fock spaces.